Liquidity provision and volatility trading

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Outline

1. What is Impermanent Loss?
2. How much is liquidity provision worth?
3. How to hedge against Impermanent Loss?
4. From liquidity provision to volatility trading

What is Impermanent Loss?

Traditional vs DeFi DEX

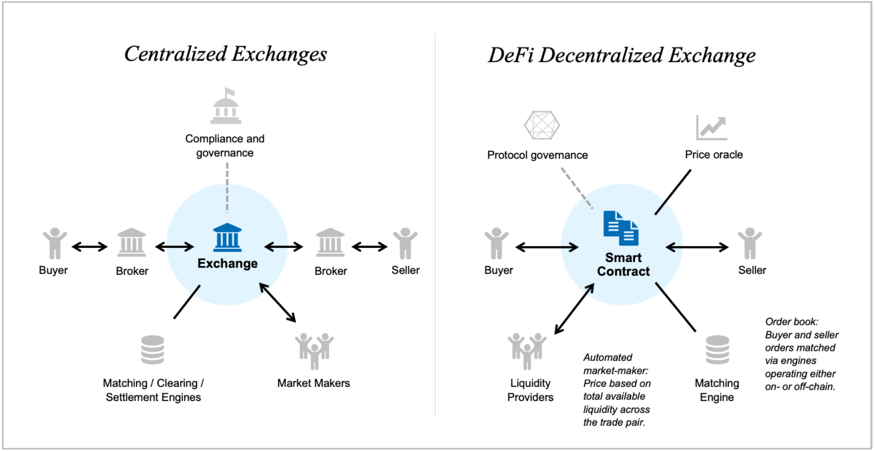

Figure: CEX vs DEX (Source: Gogel et al., "DeFi Beyond the Hype", 2021)

The Mechanics of Constant Product Market Making

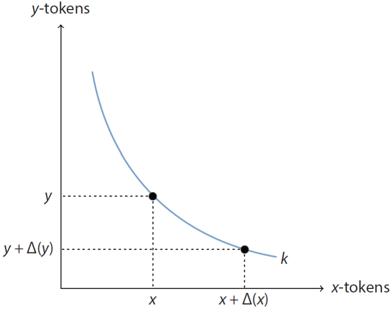

Figure: Taking & Providing Liquidity (Source: F. Schär, "Decentralized Finance", 2021)

Providing Liquidity vs HODLing

In a CPMM liquidity pool, the (infinitesimal) exchange rate is given by the ratio of reserves:

$$S_t = \frac{y_t}{x_t}.$$

Any liquidity taker's trade $(\Delta x, \Delta y)$ must leave the number L constant:

$$(x_t + \Delta x) \cdot (y_t + \Delta y) \equiv x_t \cdot y_t \equiv L^2.$$

This yields

$$x_t = \frac{L}{\sqrt{S_t}}, \quad y_t = L\sqrt{S_t}.$$

Providing Liquidity vs HODLing

What is the wealth of HODLer with initial reserves (x_0, y_0) at time $t > 0$?

$$V_{HDL}(t) = y_0 + x_0 S_t = x_0 S_0 + x_0 S_t = x_0 (S_0 + S_t)$$

What is the wealth of the liquidity provider with initial reserves (x_0, y_0) at time $t > 0$?

$$V_{LP}(0) = y_0 + x_0 S_0 = 2L\sqrt{S_0} \implies V_{LP}(t) = 2L\sqrt{S_t}$$

$$V_{LP}(t) = 2\frac{L}{\sqrt{S_0}}\sqrt{S_0 S_t}$$

$$V_{LP}(t) = 2x_0\sqrt{S_0 S_t}$$

Providing Liquidity vs HODLing

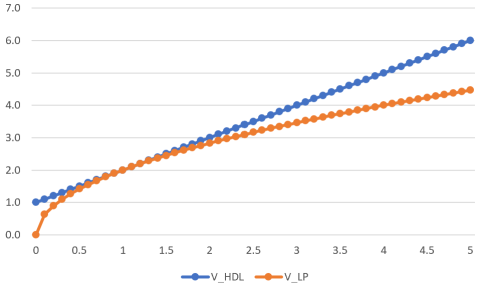

Figure: Wealth of HODLer vs liquidity provider as a function of *relative price change*

Providing Liquidity vs HODLing

Impermanent Loss is usually defined as

$$\hat{IL}(t) := \frac{V_{LP}(t) - V_{HDL}(t)}{V_{HDL}(t)}.$$

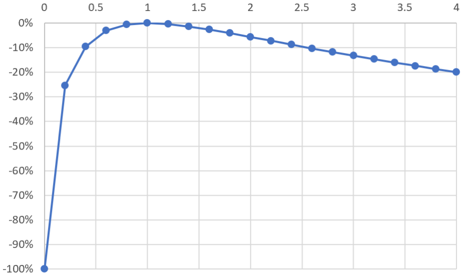

Figure: Impermanent Loss as a function of *relative price change*

Providing Liquidity vs HODLing

We prefer to work with the Impermanent Loss defined as

$$\begin{aligned} IL(t) &:= V_{HDL}(t) - V_{LP}(t) \\ &= x_0 \left(S_0 + S_t - 2\sqrt{S_0 S_t} \right) \\ &= x_0 \left(\sqrt{S_t} - \sqrt{S_0} \right)^2 \end{aligned}$$

How much is liquidity provision worth?

Risk-neutral valuation of liquidity fees

Liquidity providers in a CPMM usually earn a fixed *ad valorem* fee, e.g., 0.3% of the traded amount.
However, the actual value of liquidity provision can be inferred using risk-neutral pricing.
In a complete market, the fair price of a contingent claim hedging against Impermanent Loss is given by the present value of the corresponding expected payoff, i.e., by

$$H(t) = x_t \, \mathbb{E}_{\mathbb{Q}} \left[\left(\sqrt{S_T} - \sqrt{S_t} \right)^2 \right],$$

where the expectation is taken under the risk-neutral probability measure $\mathbb{Q}$ (and assuming that the risk-free interest rate vanishes).

Risk-neutral valuation of liquidity fees

Assume that the exchange rate S_t follows the geometric Brownian motion

$$\frac{\mathrm{d}S_t}{S_t} = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t, \quad S_0 > 0$$

for $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$, where $W = (W_t)_{t \in [0,T]}$ denotes a one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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Then the fair value of providing liquidity is given by

$$x_0 \,\mathbb{E}\left[\left(\sqrt{S_T} - \sqrt{S_0}\right)^2\right] = 2y_0 \left(1 - e^{-\sigma^2 T/8}\right).$$

How can a liquidity provider hedge against Impermanent Loss?

Connection with weighted variance swaps

Any contingent claim with a payoff function $f(S_T)$ that is twice continuously differentiable can be perfectly hedged with European calls and puts struck at a continuum of exercise prices.

The premium of such a contingent claim is given by the formula

$$f_0 = f(S_0) + \int_0^{S_0} f''(k)\ p_0(k)\ \mathrm{d}k + \int_{S_0}^{\infty} f''(k)\ c_0(k)\ \mathrm{d}k.$$

Connection with weighted variance swaps

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In our case, $f(k) = x_0 \left(\sqrt{k} - \sqrt{S_0}\right)^2$.

Therefore, the value of liquidity provision equals

$$H_0 = \frac{L}{2}\left[\int_0^{S_0} \frac{p_0(k)}{k^{3/2}}\ \mathrm{d}k + \int_{S_0}^{\infty} \frac{c_0(k)}{k^{3/2}}\ \mathrm{d}k\right].$$

Connection with weighted variance swaps

$$\int_0^T g(S_u)\,\mathrm{d}\langle \log S\rangle_u = f_g(S_T) - f_g(S_0) - \int_0^T f_g'(S_u)\,\mathrm{d}S_u.$$

Here, g is a locally integrable function, and f_g is a function satisfying

$$f_g(x) = 2\int_1^x \int_1^y \frac{g(z)}{z^2}\,\mathrm{d}z.$$

Special cases (floating legs of weighted variance swaps):

- Variance swap: $g(z) = 1 = z^0$

$$\mathbb{E}\left[1 * \langle \log(S)\rangle_T\right] = -2\,\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right]$$

Connection with weighted variance swaps

- Gamma swap: $g(z) = z = z^{1}$

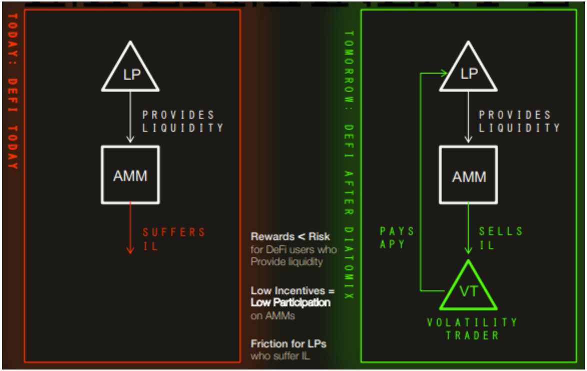
$$\frac{1}{S_0}\mathbb{E}\left[\int_0^T S_t \, \mathrm{d}\langle \log(S) \rangle_t\right] = 2\,\mathbb{E}\left[\frac{S_T}{S_0}\log\frac{S_T}{S_0}\right]$$

- Impermanent Loss: $g(z) = \sqrt{z} = z^{1/2}$

$$\mathbb{E}\left[\int_0^T \sqrt{S_t}\, \mathrm{d}\langle \log(S) \rangle_t\right] = 4\,\mathbb{E}\left[\left(\sqrt{S_T} - \sqrt{S_0}\right)^2\right].$$

It takes two to tango...

- Providing liquidity is equivalent to selling an options portfolio
- But: as of now, nobody can buy this options portfolio
- Diatomix: DeFi project that creates a marketplace for volatility
- Funding from Innosuisse (Innocheque)



Conclusions

- Impermanent Loss can be hedged in a model-free manner by using a portfolio of options with a range of different strikes

- This static hedge is closely related to the family of weighted variance swaps

- Volatility can be traded in a decentralized manner based on liquidity provision

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